CO-ARTINIAN RINGS AND MORITA DUALITY

BY

ROBERT W. MILLER AND DARRELL R. TURNIDGE

ABSTRACT

A ring A is left co-Noetherian if the injective hull of each simple left A-module is Artinian. Such rings have been studied by Vámos and Jans. Dually, call A left co-Artinian if the injective hull of each simple left A-module is Noetherian. Left co-Artinian rings having only finitely many nonisomorphic simple left modules are studied, and such rings are shown to have nilpotent radical. Moreover, it is shown that left co-Artinian implies left co-Noetherian if A/J is Artinian. For an injective left A -module ${}_AQ$ with $B = \mathrm{End}\,({}_AQ)$, and $C = \mathrm{End}\,(Q_B)$, conditions yielding a Morita duality between $\mathfrak{M}_B$ and ${}_C\mathfrak{M}$ are obtained. In special cases, e.g. ${}_AQ$ a self-cogenerator, this Morita duality yields chain conditions on ${}_AQ$. Specialized to commutative rings, these results give the known fact that every commutative co-Artinian ring is co-Noetherian. Finally in the case that the injective hull ${}_AE = E({}_AS)$ of a simple left A-module ${}_AS$ is a self-cogenerator, chain conditions on ${}_AE$ are related to chain conditions on $B_B = \mathrm{End}\,({}_AE)$. The results obtained are analogous to results for commutative rings of Vámos, Rosenberg and Zelinsky. It is shown that if A is a left co-Artinian ring with $E({}_AS)$ a self-cogenerator for each simple ${}_AS$, then J is nil and $\bigcap_{i=1}^{\infty} J^i = 0$.

A ring A is said to be left co-Noetherian if the injective hull of each simple left A-module is Artinian. Such rings have been studied by Vámos [16] and Jans [5]. Dually, A is said to be left co-Artinian if the injective hull of each simple left A-module is Noetherian. Every commutative co-Artinian ring is co-Noetherian. In §3 we investigate noncommutative left co-Artinian rings having only finitely many nonisomorphic simple left modules and show that such rings have nilpotent radical. Moreover, if A/J is Artinian, we show that left co-Artinian implies left co-Noetherian.

In §4, for an injective left A-module ${}_AQ$ with $B = \mathrm{End}({}_AQ)$, we investigate conditions yielding a Morita duality between $\mathfrak{M}_B$ and ${}_C\mathfrak{M}$, where $C = \mathrm{End}(Q_B)$. In special cases, e.g. ${}_AQ$ a self-cogenerator, this Morita duality yields chain

Received June 19, 1972

conditions on ${}_AQ$. Specialized to commutative rings, our results give the above mentioned fact that every commutative co-Artinian ring is co-Noetherian.

In §5, assuming that the injective hull ${}_AE$ of a simple left A-module ${}_AS$ is a self-cogenerator, we relate chain conditions on ${}_AE$ to chain conditions on $B_B = \mathrm{End}({}_AE)$. The results obtained are analogous to results for commutative rings of Vámos, Rosenberg, and Zelinsky. It is shown that if A is a left co-Artinian ring with $E({}_AS)$ a self-cogenerator for each simple ${}_AS$, then J is nil and $\bigcap_{i=1}^{\infty} J^i = 0$.

1. Preliminaries

Throughout this paper, A will denote an associative ring with unit and all modules will be unitary. All maps will be written on the side opposite the scalars. For an A-module X, $E(X)$ will denote the injective hull of X.

For a left A-module ${}_AQ$, Morita [9] has defined a left A-module ${}_AY$ to be of ${}_AQ$-dominant dimension $\geqq 1$ if ${}_AY$ is embeddable in a direct product of copies of ${}_AQ$. We let $\mathfrak{D}_1({}_AQ)$ denote the full subcategory of ${}_A\mathfrak{M}$ consisting of all left A-modules of ${}_AQ$-dominant dimension $\geqq 1$. It is easily seen that $Y \in \mathfrak{D}_1({}_AQ)$ if and only if the natural map $Y \to Y^{**} = \mathrm{Hom}_A(\mathrm{Hom}_B(Y, Q), Q)$ is a monomorphism where $B = \mathrm{End}({}_AQ)$ (see for example [14]). If $\mathfrak{D}_1({}_AQ) = {}_A\mathfrak{M}$, then ${}_AQ$ is said to be a cogenerator for ${}_A\mathfrak{M}$.

The following notation is due to Sandomierski [14]. For a left A-module X, let ${}_X\mathfrak{M}'$ denote the full subcategory of ${}_A\mathfrak{M}$ consisting of all left A-modules isomorphic to submodules of factor modules of X^n, $n = 1, 2, \ldots$, where X^n is a direct sum of n copies of X. A left A-module ${}_AQ$ is said to be a self-cogenerator if ${}_Q\mathfrak{M}' \subseteq \mathfrak{D}_1({}_AQ)$.

A left A-module ${}_AQ$ is said to be a ${}_AY$-injective if every A-homomorphism from a submodule ${}_AX \subseteq {}_AY$ into ${}_AQ$ extends to an A-homomorphism from ${}_AY$ into ${}_AQ$. If ${}_AQ$ is ${}_AQ$-injective, we say ${}_AQ$ is quasi-injective. Azumaya [1] has shown that if ${}_AQ$ is quasi-injective, then ${}_AQ$ is ${}_AY$-injective for all $Y \in {}_Q\mathfrak{M}'$.

LEMMA 1.1. *If ${}_AQ$ is quasi-injective, the following statements are equivalent.*

a) *${}_AX \subseteq {}_AQ$ implies $Q/X \in \mathfrak{D}_1({}_AQ)$.*

b) *${}_AQ$ is a self-cogenerator.*

PROOF.

(a) $\Rightarrow$ (b). It is sufficient to show that ${}_AX \subseteq {}_AQ^n$ implies $Q^n/X \in \mathfrak{D}_1({}_AQ)$. The proof is by induction on n. If $n = 1$, then $X \subseteq Q$ and $Q/X \in \mathfrak{D}_1({}_AQ)$ by assumption.

Let $n > 1$ and ${}_AX \subsetneqq {}_AQ^n$. Let $\rho_i\colon Q \to Q^n$ be the i'th injection map and let $\eta\colon Q^n \to Q^n/X$ be the natural map. Since $Q^n/X \neq 0$, there exists an index i such that $\phi = \rho_i \eta \neq 0$. Then $L = \operatorname{im}\phi \cong Q/\ker\phi \in \mathfrak{D}_1({}_AQ)$. Furthermore, $M = (Q^n/X)/L$ can be identified as a factor of $n-1$ copies of ${}_AQ$. Hence $M \in \mathfrak{D}_1({}_AQ)$ by the induction hypothesis. Since ${}_AQ$ is quasi-injective, we have in Fig. 1 a commutative diagram with exact rows. Since L and M are in $\mathfrak{D}_1({}_AQ)$, both α and γ are monomorphisms. A simple diagram chase shows that β is a monomorphism. Therefore, $Q^n/X \in \mathfrak{D}_1({}_AQ)$.

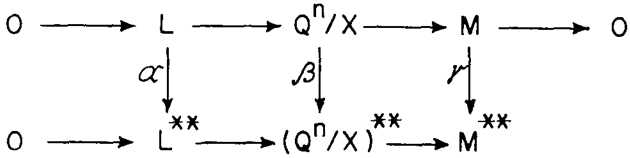

Fig. 1

(b) $\Rightarrow$ (a). Trivial.

Let ${}_AQ$ be a left A-module with $B = \operatorname{End}({}_AQ)$. For ${}_AX$ a submodule of ${}_AY$ and L_B a submodule of $Y^*_B = \operatorname{Hom}_A(Y, Q)$, let $\operatorname{Ann}_{Y^*}(X) = \{f \in Y^* \mid Xf = 0\}$ and $\operatorname{Ann}_Y(L) = \{y \in Y \mid yL = 0\}$. The following lemma is left to the reader.

LEMMA 1.2. *For* ${}_AX \subseteq {}_AY$, $\operatorname{Ann}_{Y^*}(X)_B \cong \operatorname{Hom}_A(Y/X, Q)_B$.

Let A be a ring with Jacobson radical J (see [6]). If given any sequence $\{x_i\}_{i=1}^{\infty} \subseteq J$ there is a finite index n such that $x_1 x_2 \dots x_n = 0$, then J is said to be left T-nilpotent. Bass [2] has defined a ring A to be left (right) perfect if A/J is Artinian and J is left (right) T-nilpotent. If A/J is Artinian and J is nilpotent, then A is said to be semiprimary.

Given a left A-module X, the socle of ${}_AX$, denoted So $({}_AX)$, is defined to be the sum of all of the simple submodules of ${}_AX$. If ${}_AX$ has no simple submodules, So $({}_AX) = 0$. Bass [2] has shown that a ring A is left perfect if and only if A/J is Artinian and So (M_A) is an essential submodule of M_A for all $M \in \mathfrak{M}_A$.

A left A-module ${}_AX$ is said to be cofinitely generated if ${}_AX$ has finitely generated essential socle (see [5] and [16]). Morita [9] has called ${}_AX$ finitely cogenerating if $A/\operatorname{Ann}_A(X)$ can be embedded as a left A-module in a finite direct sum of copies of ${}_AX$.

Let ${}_AQ$ be an injective left A-module. Then $\mathfrak{D}_1({}_AQ)$ is closed under submodules, extensions, arbitrary direct products, and injective hulls. Thus $\mathfrak{D}_1({}_AQ)$ is a torsion-free class as defined by Dickson [4] with

$$\mathfrak{I}({}_AQ) = \{X \in {}_A\mathfrak{M} \mid \mathrm{Hom}_A(X, Q) = 0\}$$

as its associated hereditary torsion class. We let $A_{\mathfrak{I}}$ denote the ring of left quotients of A with respect to the hereditary torsion class $\mathfrak{I}({}_AQ)$. By [9, Th. 5.6], if ${}_AQ$ is finitely cogenerating, then $A_{\mathfrak{I}}$ coincides with $C = \mathrm{End}(Q_B)$ where $B = \mathrm{End}({}_AQ)$. The reader is referred to [7] and [15] for further information on torsion theories and rings of quotients.

Let ${}_AQ$ be an injective cogenerator for ${}_A\mathfrak{M}$ with $B = \mathrm{End}({}_AQ)$. The bimodule ${}_AQ_B$ is said to afford a Morita duality between ${}_A\mathfrak{M}$ and $\mathfrak{M}_B$ if the natural maps

$$ {}_AX \to {}_AX^{**} = \mathrm{Hom}_B(\mathrm{Hom}_A(X, Q), Q)$$

$$L_B \to L_B^{**} = \mathrm{Hom}_A(\mathrm{Hom}_B(L, Q), Q)$$

are isomorphisms for all $X \in {}_A\mathfrak{M}'$ and all $L \in \mathfrak{M}'_B$. It is well known that an injective cogenerator ${}_AQ$ yields such a Morita duality if and only if Q_B is an injective cogenerator and $A \cong \mathrm{End}(Q_B)$. The reader is referred to [10], [11], and [12].

2. Closed submodules

Throughout this section ${}_AQ$ will denote a quasi-injective left A-module with A-endomorphism ring B. For ${}_AY$, we examine an order reversing correspondence between A-submodules of Y and B-submodules of its Q-dual $Y^* = \mathrm{Hom}_A(Y, Q)$. If $Y = Q$, this correspondence relates finiteness conditions on the ring B to certain chain conditions on ${}_AQ$.

Let ${}_AX$ be a submodule of ${}_AY$. We call X a closed submodule of Y with respect to ${}_AQ$ if Y/X can be embedded in a direct product of copies of ${}_AQ$, i.e. $Y/X \in \mathfrak{D}_1({}_AQ)$. If Y/X can be embedded in a finite direct sum of copies of ${}_AQ$, we call X a finitely closed submodule of Y with respect to ${}_AQ$. In the case that ${}_AQ$ is injective, $\mathfrak{D}_1({}_AQ)$ is a torsion-free class, and X is a closed submodule of Y with respect to ${}_AQ$ if and only if Y/X is torsion free [7, p. 3]. When there is no possibility of confusion, we will simply call ${}_AX$ a (finitely) closed submodule of ${}_AY$.

The following theorem is essentially in [14]. We include the proof for completeness.

THEOREM 2.1. *Let ${}_AQ$ be quasi-injective with $B = \mathrm{End}({}_AQ)$.*

a) *If ${}_AX$ is a closed submodule of ${}_AY$, then $\mathrm{Ann}_Y\mathrm{Ann}_{Y^*}(X) = X$.*

b) *There is a one-to-one order reversing correspondence between the finitely* *closed submodules of* ${}_AY$ *and the finitely generated submodules of* Y^*_B *given by* ${}_AX \to \mathrm{Ann}_{Y^*}(X)$. *The inverse correspondence is given by* $L_B \to \mathrm{Ann}_Y(L)$.

PROOF.

a) Let X be a closed submodule of Y. Since $Y/X \in \mathfrak{D}_1({}_AQ)$, there exists $\{f_i \mid f_i \in Y^*\}_{i\in I}$ such that $\cap_{i\in I} \mathrm{Ker} f_i = X$. Clearly $X \subseteq \mathrm{Ann}_Y \mathrm{Ann}_{Y^*}(X) = \cap \{\mathrm{Ker} f \mid f \in Y^*, X \subseteq \mathrm{Ker} f\} \subseteq \cap_{i\in I} \mathrm{Ker} f_i = X$.

b) Let ${}_AX$ be a finitely closed submodule of Y. Thus there exists an exact sequence

$$0 \to Y/X \to {}_AQ^n.$$

Since ${}_AQ$ is quasi-injective,

$$\mathrm{Hom}_A(Q^n, Q)_B \to \mathrm{Hom}_A(Y/X, Q)_B \to 0$$

is exact. Thus $\mathrm{Ann}_{Y^*}(X) \cong \mathrm{Hom}_A(Y/X, Q)$ is finitely generated since $\mathrm{Hom}_A(Q^n, Q) \cong B^n_B$. That $X = \mathrm{Ann}_Y \mathrm{Ann}_{Y^*}(X)$ follows by (a).

Let $L_B = f_1B + \dots + f_nB$ be a finitely generated submodule of Y^*_B. Since $\cap_{i=1}^n \mathrm{Ker} f_i = \cap_{f\in L} \mathrm{Ker} f = \mathrm{Ann}_Y(L) \subseteq \mathrm{Ker} f_i$, each f_i induces a map $\bar{f}_i \colon Y/\mathrm{Ann}_Y(L) \to Q$. Thus we have the exact sequence

$$0 \to Y/\mathrm{Ann}_Y(L) \xrightarrow{\phi} Q^n$$

where $\phi = (\bar{f}_1, \dots, \bar{f}_n)$. Thus $\mathrm{Ann}_Y(L)$ is a finitely closed submodule of ${}_AY$.

Clearly $L_B \subseteq \mathrm{Ann}_{Y^*}\mathrm{Ann}_Y(L)$. Let $g \in \mathrm{Ann}_{Y^*}\mathrm{Ann}_Y(L)$. Then g induces a map $\bar{g} \colon Y/\mathrm{Ann}_Y(L) \to Q$. We have in Fig. 2 a commutative diagram where $\lambda \in \mathrm{Hom}_A(Q^n, Q) \cong B^n$ follows by the quasi-injectivity of ${}_AQ$. Thus $\bar{g} = \phi\lambda = \sum_{i=1}^n \bar{f}_i b_i$, and so $g = \sum_{i=1}^n f_i b_i \in L_B$.

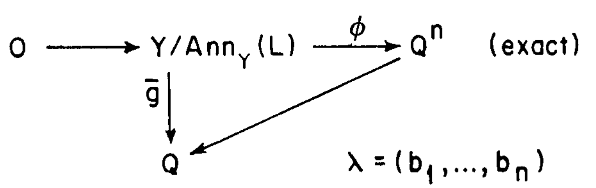

Fig. 2

If ${}_AY = {}_AQ$, we have the following corollary.

COROLLARY 2.2. *Let* ${}_AQ$ *be quasi-injective with* $B = \mathrm{End}({}_AQ)$. *Then*

a) *B is right Noetherian if and only if $_AQ$ has* DCC *on finitely closed submodules.*

b) *B is left perfect if and only if $_AQ$ has* ACC *on finitely closed submodules.*

c) *B is semiprimary if $_AQ$ has* ACC *on closed submodules.*

PROOF.

a) Trivial.

b) This follows from the fact that a ring is left perfect if and only if it has DCC on finitely generated right ideals [3, Th. 2].

c) By (b), it is sufficient to show that the radical N of B is nilpotent. Since B/N is semisimple, $\mathrm{So}(W_B) = \mathrm{Ann}_W(N)$ for all $W \in \mathfrak{M}_B$. Let $\mathrm{So}^1(Q_B) = \mathrm{So}(Q_B)$ and for $i > 1$, inductively define $\mathrm{So}^i(Q_B)$ to be the inverse image in Q_B of $\mathrm{So}(Q/\mathrm{So}^{i-1}(Q))$. It is easy to see that $\mathrm{So}^i(Q_B) = \mathrm{Ann}_Q(N^i)$, hence $\mathrm{So}^i(Q_B)$ is an A-submodule of Q. Also $\mathrm{So}^i(Q_B)$ is a closed A-submodule of Q since there is a natural A-embedding of $Q/\mathrm{Ann}_Q(N^i)$ into a direct product of copies of $_AQ$. Since B is left perfect, $\mathrm{So}^{i-1}(Q_B) \subsetneqq \mathrm{So}^i(Q_B)$ unless $\mathrm{So}^{i-1}(Q_B) = Q$. By hypothesis, $\mathrm{Ann}_Q(N^i) = \mathrm{So}^i(Q_B) = Q$ for some integer i. Thus $N^i = 0$ since Q_B is faithful.

3. Co-Artinian rings

Jans [5] has called a ring A left co-Noetherian if factors of cofinitely generated left A-modules are cofinitely generated. Using Vámos' results, Jans shows that A is left co-Noetherian if and only if the injective hull of each simple left A-module is Artinian. This is equivalent to property (P) of Vámos [16] that every cofinitely generated left A-module is Artinian.

Dually, we call a ring A left co-Artinian if the injective hull of each simple left A-module is Noetherian. It is easily seen that this definition is equivalent to property (Q) of Vámos which requires that every cofinitely generated left A-module be finitely generated. Compare this with Vámos' result [16, Proposition 5] that A is left Artinian if and only if every finitely generated left A-module is cofinitely generated. One easily sees that a ring A having only a finite number of non-isomorphic simple left A-modules is left co-Noetherian (co-Artinian) if and only if the minimal injective cogenerator for $_A\mathfrak{M}$ is Artinian (Noetherian).

In this section, we relate the co-Artinian and co-Noetherian properties for rings A with A/J Artinian, and extend a result of Rosenberg and Zelinsky [13, Th. 4] which shows that J is nilpotent if the minimal injective cogenerator for $_A\mathfrak{M}$ has finite length. Our proofs rely heavily upon the techniques of Rosenberg and Zelinsky.

LEMMA 3.1. *Let $_AQ$ be the minimal injective cogenerator for $_A\mathfrak{M}$. If $_AQ$ is Noetherian (Artinian) then A has* DCC (ACC) *on two-sided ideals.*

PROOF. We consider only the case when $_AQ$ is Noetherian as the Artinian case follows in a similar manner.

Let $A \supseteq L_1 \supseteq L_2 \supseteq \dots$ be a descending chain of two-sided ideals of A. Let $Q_i = \mathrm{Ann}_Q(L_i)$. Then we have $0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots$. Since $_AQ$ is Noetherian, there exists an integer k such that $Q_k = Q_{k+j}$ for all $j \geqq 1$. By [13, Lemma 1]

$$\mathrm{Hom}_A(L_k/L_{k+j}, Q) \cong Q_{k+j}/Q_k = 0$$

for all $j \geqq 1$. Thus $L_k = L_{k+j}$ for all $j \geqq 1$ since $_AQ$ is a cogenerator.

THEOREM 3.2. *If the minimal injective cogenerator $_AQ$ for $_A\mathfrak{M}$ is Noetherian, then J is nilpotent.*

PROOF. Consider the descending chain $A \supseteq J \supseteq J^2 \supseteq \dots$ of two-sided ideals. By Lemma 3.1, there exists an integer k such that $J^k = J^{k+1}$. Then $J^kQ = J^{k+1}Q$. Since J^kQ is finitely generated, this forces $J^kQ = 0$ by Nakayama's lemma. Thus $J^k = 0$ since $_AQ$ is faithful (see [13, Lemma 6]).

In view of the previously mentioned result of Rosenberg and Zelinsky, one might ask whether or not the minimal injective cogenerator for $_A\mathfrak{M}$ has finite length whenever it is Noetherian. For commutative rings, this is the case since any commutative co-Artinian ring is co-Noetherian (see [16, Prop. 4] or our Corollary 4.7). The authors have been unable to answer this question in general but have obtained the following partial solution.

PROPOSITION 3.3. *Let A/J be Artinian and let $_AQ$ be an injective cogenerator for $_A\mathfrak{M}$. The following statements are equivalent.*

a) *$_AQ$ is Noetherian.*

b) *$_AQ$ is Artinian and J is nilpotent.*

c) *$_AQ$ is Artinian and J is left T-nilpotent.*

PROOF. If $_AQ$ is Noetherian, then J is nilpotent by Theorem 3.2. Thus given either (a) or (b) we have a descending chain

$$Q \supsetneqq JQ \supsetneqq J^2Q \supsetneqq \cdots \supsetneqq J^kQ = 0$$

for some integer k. A similar chain follows by (c) since $J^{i+1}Q \subsetneqq J^iQ$ if $J^iQ \neq$ by the left T-nilpotence of J. Now if ${}_AQ$ is Noetherian (Artinian), then each $J^iQ/J^{i+1}Q$ is semisimple Noetherian (Artinian) as an A-module. In any case, the above chain can be refined to a composition series.

COROLLARY 3.4. *For A/J Artinian, the following statements are equivalent.*

a) *A is left co-Artinian.*

b) *A is left co-Noetherian and semiprimary.*

c) *A is left co-Noetherian and left perfect.*

REMARKS.

1) The localization of the integers at a prime p yields an example of a commutative, local (hence semiperfect) co-Noetherian ring which is not co-Artinian. See [13, Lemma 7].

2) Corollary 3.4 raises the question of whether or not a left co-Noetherian right perfect ring is left co-Artinian.

3) It is possible that any ring satisfying the equivalent conditions of Corollary 3.4 is left Artinian. Mueller observes [10, p. 1344] that Osofsky's conjecture (P) [12, p. 385] implies that a semiperfect ring A has a left Morita duality whenever the minimal injective cogenerator ${}_AQ$ for ${}_A\mathfrak{M}$ has finite length. In the setting of Corollary 3.4, ${}_AQ$ has finite length and A is semiprimary; thus conjecture (P) would imply that A is left Artinian [12, Th. 3].

4. Duality

Vámos [16, Prop. 4] has shown that every cofinitely generated Noetherian module over a commutative ring A is Artinian. Thus, as noted in §3, every commutative co-Artinian ring is co-Noetherian. Vámos' result is a consequence of a theorem of Matlis [8, Prop. 3] that every cofinitely generated module over a commutative Noetherian ring is Artinian. The proof of Matlis' result essentially depends upon the fact that the localization A_M of a commutative ring A with respect to a maximal ideal M has Morita duality if the injective hull $E(A/M)$ is Noetherian. In the following we investigate an analogous situation in the noncommutative case.

Let $\mathfrak{I}$ be a hereditary torsion class for ${}_A\mathfrak{M}$ and let $A_\mathfrak{I}$ be the ring of left quotients of A with respect to $\mathfrak{I}$. Via the natural ring homomorphism $A \to A_{\mathfrak{I}}$, each $A_{\mathfrak{I}}$-module can be viewed as an A-module. We call $\mathfrak{I}$ a perfect torsion class (compare [15, exercise 1, p. 81]) if every left $A_{\mathfrak{I}}$-module is torsion free viewed as an A-module. Any hereditary torsion class $\mathfrak{I}$ is of the form $\mathfrak{I} = \mathfrak{I}({}_AE) = \{X \in {}_A\mathfrak{M} \mid \mathrm{Hom}_A(X, E) = 0\}$ for some finitely cogenerating injective left A-module ${}_AE$. Moreover, Morita has shown [9, Th. 5.6] that $A_{\mathfrak{I}}$ coincides with the double centralizer of any finitely cogenerating injective A-module ${}_AE$ which determines $\mathfrak{I}$. Throughout the following, ${}_AE$ denotes an injective A-module with $B = \mathrm{End}({}_AE)$ and $C = \mathrm{End}(E_B)$.

LEMMA 4.1. *Let ${}_AE$ be a finitely cogenerating injective A-module. Then $\mathfrak{I}({}_AE)$ is a perfect torsion class if and only if ${}_CE$ is an injective cogenerator for ${}_C\mathfrak{M}$.*

PROOF. Suppose that $\mathfrak{I}({}_AE)$ is perfect. By [9, Th. 2.3] ${}_CE$ is injective and ${}_CE \cong {}_C\mathrm{Hom}_A({}_AC_C, {}_AE)$. Thus for $X \in {}_C\mathfrak{M}$

$$\mathrm{Hom}_C(X, E) \cong \mathrm{Hom}_C(X, \mathrm{Hom}_A(C, E)) \cong \mathrm{Hom}_A({}_AC \otimes {}_CX, E) \cong \mathrm{Hom}_A(X, E).$$

In particular, let X be a simple left C-module. Since $\mathfrak{I}({}_AE)$ is perfect, $X \in \mathfrak{D}_1({}_AE)$ hence $\mathrm{Hom}_A(X, E) \neq 0$. Thus $\mathrm{Hom}_C(X, E) \neq 0$, and therefore, ${}_CE$ is an injective cogenerator.

Conversely, let ${}_CE$ be an injective cogenerator and let $X \in {}_C\mathfrak{M}$. Then $X \in \mathfrak{D}_1({}_CE)$, hence as an A-module, $X \in \mathfrak{D}_1({}_AE)$. That is, ${}_AX$ is torsion free.

In the commutative case we obtain the following consequence of Lemma 4.1.

COROLLARY 4.2. *Let A be a commutative ring with maximal ideal M. If ${}_AE = E(A/M)$ is finitely cogenerating (e.g. if ${}_AE$ is Noetherian), then $\mathfrak{I}({}_AE)$ is perfect and C coincides with the localization A_M of A with respect to the maximal ideal M.*

PROOF. Since ${}_AE$ is finitely cogenerating, C coincides with $A_{\mathfrak{I}} = A_M$. By [13, Lemma 7], as an A_M-module, E is an injective cogenerator.

THEOREM 4.3. *Let ${}_AE$ be injective with $B = \mathrm{End}({}_AE)$ and $C = \mathrm{End}(E_B)$. If E_B is finitely cogenerating (e.g. if ${}_AE$ is finitely generated) and Noetherian and if $\mathfrak{I}({}_AE)$ is perfect, then*

a) *B_B and ${}_CC$ are Artinian.*

b) *E_B and ${}_CE$ are injective cogenerators of finite length.*

Hence ${}_CE_B$ yields a Morita duality between $\mathfrak{M}_B$ and ${}_C\mathfrak{M}$.

PROOF. Since E_B is finitely generated, ${}_AE$ is finitely cogenerating.Thus by Lemma 4.1, ${}_CE$ is an injective cogenerator since $\mathfrak{J}({}_AE)$ is perfect. Now $B = \mathrm{End}({}_CE)$ and $E_B \cong \mathrm{Hom}_C({}_CC, {}_CE_B)$ is the ${}_CE$-dual of ${}_CC$. Every submodule of ${}_CC$ is closed with respect to ${}_CE$ since ${}_CE$ is a cogenerator. Thus ${}_CC$ is Artinian by (a) of Theorem 2.1 since E_B is Noetherian.

Since E_B is finitely cogenerating and faithful, there exists an exact sequence

$$0 \to B_B \xrightarrow{i} E^n_B.$$

Applying the functor ${}_C\mathrm{Hom}_B(\ \ , E_B)$ yields the exact sequence

$${}_CC^n \xrightarrow{\hat{\imath}} {}_CE \to {}_CW \to 0$$

where $W = \mathrm{coker}\,\hat{\imath}$. Since ${}_CE$ is injective, we have in Fig. 3 a commutative diagram with exact rows where the vertical maps are isomorphisms. Thus $\mathrm{Hom}_C(W, E) = 0$ which implies that $W = 0$ since ${}_CE$ is a cogenerator. Therefore, ${}_CE$ is finitely generated.

$$\begin{array}{ccccccc} 0 & \longrightarrow & & & B_B & \longrightarrow & E^n_B \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_C(W,E) & \longrightarrow & \mathrm{Hom}_C(E,E) & \longrightarrow & \mathrm{Hom}_C(C^n,E) \end{array}$$

Fig. 3

Since ${}_CC$ is Artinian, ${}_CE$ being finitely generated implies that ${}_CE$ has finite length. Thus E_B is an injective cogenerator (see for example Lemmas 3.5 and 3.7 of [14]). Hence ${}_CE_B$ yields a Morita duality between $\mathfrak{M}_B$ and ${}_C\mathfrak{M}$.

By (c) of Corollary 2.2, B is semiprimary, so by [12, Th. 3] B_B is Artinian. Finally, E_B is finitely generated and thus has finite length.

REMARK. Suppose further in Theorem 4.3 that ${}_AE$ is cofinitely generated with $\mathrm{So}({}_AE) = S = \oplus_{i=1}^n S_i$ where each ${}_AS_i$ is simple. For each ${}_AX \subseteq {}_AE$, there are natural left C-isomorphisms $L_{\mathfrak{J}}(X) \cong C \otimes_A X \cong CX$ where $L_{\mathfrak{J}}(X)$ denotes the localization of X with respect to $\mathfrak{J} = \mathfrak{J}({}_AE)$. The first isomorphism follows since $\mathfrak{J}$ is perfect [15, Th. 13.1]. The second is verified by noting that the composition of the natural epimorphism $C \otimes_A X \to CX$ with the essential left A-monomorphism $X \to L_{\mathfrak{J}}(X) \cong C \otimes_A X$ [15, Propositions 6.4 and 8.1] is a monomorphism. In particular, each ${}_AS_i$ is essential in ${}_ACS_i$, so ${}_CCS_i$ is simple as a C-module. As left C-modules, $CS \cong C \otimes_A S \cong C \otimes_A (\oplus_{i=1}^n S_i) \cong \oplus_{i=1}^n (C \otimes_A S_i) \cong \oplus_{i=1}^n CS_i$. Clearly ${}_CCS$ is essential in ${}_CE$, thus ${}_CCS = \mathrm{So}({}_CE)$ is finitely generated and essential

More specifically, if ${}_AE = E({}_AS)$ is the injective hull of a simple left A-module ${}_AS$, then B_B is local by [6, Exercise 3, p. 104]. Also, ${}_CE$ has essential simple socle ${}_CCS$ by the above. Since ${}_CE$ is an injective cogenerator, ${}_CCS$ is the unique (up to isomorphism) simple left C-module.

In the following, we obtain information on chain conditions for ${}_AE$ by assuming that ${}_AE$ is a self cogenerator in Theorem 4.3.

LEMMA 4.4. *Let ${}_AE$ be a finitely cogenerating injective left A-module. The following statements are equivalent.*

a) *${}_AE$ is a self-cogenerator.*

b) *$\mathfrak{J}({}_AE)$ is perfect and every A-submodule of E is a C-submodule.*

c) *$\mathfrak{J}({}_AE)$ is perfect and $C = \bar{A}$ the canonical image of A in C.*

PROOF.

(a) $\Rightarrow$ (c). As before, ${}_CE$ is injective and finitely cogenerating by [9, Th. 2.3]. We identify ${}_CC$ as a submodule of ${}_CE^n$ for some integer n. Let M be a maximal left ideal of C. Then ${}_C(C/M) \subseteq {}_C(E^n/M)$. Since ${}_AE$ is a self-cogenerator, ${}_A(C/M) \in \mathfrak{D}_1({}_AE)$. Hence $0 \neq \mathrm{Hom}_A(C/M, E) = \mathrm{Hom}_C(C/M, E)$. So ${}_CE$ contains a copy of each simple left C-module, hence is an injective cogenerator. By Lemma 4.1, $\mathfrak{J}({}_AE)$ is perfect.

As a left A-module, $C/\bar{A}$ is torsion [15, Lemma 7.5]. As above, since ${}_AE$ is a self-cogenerator, $C/\bar{A} \in \mathfrak{D}_1({}_AE)$, hence is torsion free. Thus $C = \bar{A}$ since $C/\bar{A} = 0$.

(c) $\Rightarrow$ (b). This is easy since $\bar{A} \cong A/\mathrm{Ann}_A(E)$.

(b) $\Rightarrow$ (a). Let ${}_AK \subseteq {}_AE$. Since $\mathfrak{J}({}_AE)$ is perfect, ${}_CE$ is an injective cogenerator. Thus ${}_C(E/K) \in \mathfrak{D}_1({}_CE)$ since K is a C-submodule of E. Hence ${}_A(E/K) \in \mathfrak{D}_1({}_AE)$. Therefore, (a) follows by Lemma 1.1.

PROPOSITION 4.5. *Let ${}_AE = E({}_AS)$ be the injective hull of a simple left A-module ${}_AS$. Suppose ${}_AE$ satisfies the hypotheses of Theorem 4.3. Then*

a) ${}_AE$ *has finite length if* ${}_AE$ *is a self-cogenerator.*

b) ${}_AE$ *is a self-cogenerator if* ${}_CC$ *is local.*

PROOF.

a) By Theorem 4.3, ${}_CE$ has finite length. But every A-submodule of E is a C-submodule by Lemma 4.4. Hence ${}_AE$ has finite length.

b) If ${}_CC$ is local, then E_B has essential simple socle by the Morita duality. Since S is a B-submodule of E, $\mathrm{So}(E_B) \subseteq S_B$. By the remark following Theorem 4.3, $\mathrm{So}({}_CE) = CS$. Thus $\mathrm{Ann}_B(CS) = N$, the radical of B. Therefore, $S \subseteq CS \subseteq \mathrm{Ann}_E\mathrm{Ann}_B(CS) = \mathrm{Ann}_E(N) = \mathrm{So}(E_B)$ since B/N is semisimple. Hence $S = \mathrm{So}(E_B) = \mathrm{So}({}_CE) = CS$.

Let ${}_CE \supset X_1 \supset X_2 \supset \cdots \supset X_n = 0$ be a C-composition series for E. This is also an A-composition series since each of the factors is isomorphic to $CS = S$. If ${}_AK \subseteq {}_AE$, then E/K is cofinitely generated since ${}_AE$ is Artinian [16, Proposition 5]. Then $E/K \in \mathfrak{D}_1({}_AE)$ since $\mathrm{So}(E/K) \cong S^m$ for some integer m. Therefore, (b) follows by Lemma 1.1.

Applying the preceding to the commutative case yields the following known result.

PROPOSITION 4.6. *Let A be a commutative ring with maximal ideal M. If ${}_AE = E(A/M)$ is Noetherian, then E_B is Noetherian and $\mathfrak{I}({}_AE)$ is perfect. Furthermore, $C \cong A_M$ is local, hence ${}_AE$ is a self-cogenerator of finite length.*

PROOF. Since A is commutative, there is a natural ring homomorphism $A \to B$, hence every B-submodule of E is an A-submodule. Thus E_B is Noetherian. Since $\mathfrak{I}({}_AE)$ is perfect by Corollary 4.2, the result follows by Theorem 4.3 and Proposition 4.5.

COROLLARY 4.7. *Every commutative co-Artinian ring is co-Noetherian.*

5. Self-cogenerators

If A is a commutative co-Artinian ring, then $E({}_AS)$ is an Artinian self-cogenerator for every simple A-module ${}_AS$. For any ring A, $E({}_AS)$ has finite length whenever it is a self-cogenerator satisfying the hypotheses of Theorem 4.3. In this section, we give a more direct method of obtaining chain conditions on $E({}_AS)$ when it is a self-cogenerator.

PROPOSITION 5.1. *For ${}_AS$ simple, let ${}_AE = E({}_AS)$ be a self-cogenerator with $B = \mathrm{End}({}_AE)$. Then ${}_AE$ is Artinian if and only if B_B is Noetherian.*

PROOF. If ${}_AE$ is Artinian, then B_B is Noetherian by (a) of Corollary 2.2. Conversely, let B_B be Noetherian. Since ${}_AE$ is a self-cogenerator, every submodule of ${}_AE$ is closed with respect to ${}_AE$. Hence ${}_AE$ is Artinian by (a) of Theorem 2.1.

REMARK. Let A be a commutative ring with maximal ideal M. Compare Proposition 5.1 with [16, Th. 2] that ${}_AE = E(A/M)$ is Artinian if and only if A_M is Noetherian. By our results in §4, if ${}_AE$ is Noetherian, then A_M has Morita duality with $B = \mathrm{End}({}_AE)$. Since ${}_{A_M}E$ is the minimal injective cogenerator for ${}_{A_M}\mathfrak{M}$, $B \cong A_M$ [11, Th. 3]. Rosenberg and Zelinsky have shown [13, Th. 5] that ${}_AE = E(A/M)$ has finite length if and only if A_M is Artinian. The following proposition gives an analogous result for noncommutative rings.

PROPOSITION 5.2. *For ${}_AS$ simple, let ${}_AE = E({}_AS)$ be a self-cogenerator with $B = \mathrm{End}({}_AE)$. The following statements are equivalent.*

a) *${}_AE$ has finite length.*

b) *${}_AE$ is Noetherian and ${}_A(E/S)$ is cofinitely generated.*

c) *B_B is Artinian.*

PROOF.

(a) $\Rightarrow$ (b). Clear.

(b) $\Rightarrow$ (c). B is semiprimary by (c) of Corollary 2.2. Since ${}_A(E/S)$ is cofinitely generated and ${}_AE$ is a self-cogenerator, ${}_AS$ is a finitely closed submodule of ${}_AE$. Thus by (b) of Theorem 2.1, $N_B = \mathrm{Ann}_B(S)$ the radical of B is finitely generated. Hence B_B is Artinian by [12, Lemma 11].

(c) $\Rightarrow$ (a). If B_B is Artinian, then B_B has finite length. Since ${}_AE$ is a self-cogenerator, every submodule of ${}_AE$ is closed with respect to ${}_AE$. Hence ${}_AE$ has finite length by (a) of Theorem 2.1.

By [13, Th. 4], if the injective hull of each simple left A-module has finite length then

1) J is nil and $\cap_{i=1}^{\infty} J^i = 0$.

2) J is nilpotent if A has only finitely many nonisomorphic simple left modules.

In Theorem 3.2, we observed that (2) remains true assuming only that the injective hull of each simple left A-module is Noetherian, i.e. A is left co-Artinian. The following theorem shows that (1) is valid for any left co-Artinian ring having $E({}_AS)$ a self-cogenerator for each simple left A-module ${}_AS$.

THEOREM 5.3. *Let A be a left co-Artinian ring having $E(_AS)$ a self-cogenerator for each simple left A-module $_AS$. Then J is nil and* $\bigcap_{i=1}^{\infty} J^i = 0$.

PROOF. Let $_AS$ be a simple left A-module with $_AE = E(_AS)$. Setting $_AE_i = \mathrm{Ann}_E(J^i)$, we have $0 \subseteq E_1 \subseteq E_2 \subseteq \ldots$. By hypothesis, $_AE$ is Noetherian, hence there exists an integer k such that $E_k = E_{k+1}$. By [13, Lemma 1]

$$\mathrm{Hom}_A(J^k/J^{k+1}, E) \cong E_{k+1}/E_k = 0.$$

Thus

$$0 = \mathrm{Hom}_A(E, \mathrm{Hom}_A(J^k/J^{k+1}, E) = \mathrm{Hom}_A(J^k/J^{k+1} \otimes {}_AE, E).$$

There is a natural left A-epimorphism

$$J^k/J^{k+1} \otimes {}_AE \to J^kE/J^{k+1}E.$$

Applying the functor $\mathrm{Hom}_A(\quad, {}_AE)$ we have $\mathrm{Hom}_A(J^kE/J^{k+1}E, E) = 0$. Thus $J^kE/J^{k+1}E = 0$ since $_AE$ is a self-cogenerator. Since $_AE$ is Noetherian and $J^kE = J^{k+1}E$, we see that $J^kE = 0$. Thus, some power of J annihilates the injective hull of each simple left A-module.

The remainder of the proof is identical to the proof of [13, Th. 4].

ACKNOWLEDGEMENT

The authors wish to thank Professor F. L. Sandomierski for his helpful discussions.

REFERENCES

1. G. Azumaya, *M-projective and M-injective modules*, Ring Theory Symposium Notes, University of Kentucky, May 1970.

2. H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

3. J. Björk, *Rings satisfying a minimum condition on principal ideals*, Crelles J. **236** (1969), 112–144.

4. S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. **121** (1966), 233–235.

5. J. P. Jans, *On co-Noetherian rings*, J. London Math. Soc. (2) **1** (1969), 588–590.

6. J. Lambek, *Lectures on Rings and Modules*, Waltham, Mass., 1966.

7. J. Lambek, *Torsion theories, additive semantics, and rings of quotients*, Springer Lecture Notes 177, 1971.

8. E. MATLIS, *Modules with descending chain condition*, Trans. Amer. Math. Soc. **97** (1960), 495–508.

9. K. MORITA, *Localizations in categories of modules. I*, Math. Z. **114** (1970), 121–144.

10. B. MUELLER, *On Morita duality*, Canad. J. Math. **21** (1969), 1338–1347.

11. B. J. Mueller, *Linear compactness and Morita duality*, J. Algebra **16** (1970), 60–66.

12. B. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373–387.

13. A. Rosenberg and D. Zelinsky, *Finiteness of the injective hull*, Math. Z. **70** (1959) 372–380.

14. F. Sandomierski, *Linearly compact modules and local Morita duality*, Proceedings of the Ring Theory Conference, Salt Lake City, Utah, March 1971.

15. B. Stenstrom, *Rings and modules of quotients*, Springer Lecture Notes 237, 1971.

16. P. Vamos, *The dual of the notion of "finitely generated"*, J. London Math. Soc. **43** (1968), 643–646.

Department of Mathematics
College of William and Mary
Williamsburg, Virginia, U.S.A.

and

Department of Mathematics
Kent State University
Kent, Ohio, U.S.A.